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SOME NEW QUANTUM INEQUALITIES VIA tgs-CONVEX FUNCTIONS

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Abstract. The objective of this paper is to obtain some new integral inequalities via tgs-convex functions in the setting of quantum calculus. We link our results with the ordinary calculus case by discussing special cases of our main results.

Keywords: convex, tgs-convex, functions, quantum, q-differentiable, integral inequalities

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1. Introduction and preliminaries

It is well known that the convexity theory has played a fundamental part in the developments of various branches of pure and applied sciences. Due to its numerous applications, convex functions have been generalized and extended in various directions using novel and innovative ideas. It have been shown that the optimality conditions of the differentiable convex functions can be characterized by variational inequalities, which have been used to study complicated and difficult problems. For the numerical methods, formulation, generalizations and other aspects of convex functions, see [1, 3, 8, 13, 23, 24] and the references therein. Inspired by this, Tunc et al. [23] introduced the notion of so-called tgs-convex functions as:

Definition 1.1 ([23]). *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a non-negative function. We say that f is tgs-convex function on I, if*

$$f((1-t)u+tv) \leq t(1-t)[f(u)+f(v)], \quad \forall u, v \in I, t \in]0,1[. \tag{1}$$

Note that f is tgs-concave function if $-f$ is tgs-convex function. Also if $t = 0, 1$, then, according to the hypothesis the function is equal to zero.

Remark 1.1. *It has been noticed that the class of tgs-convex functions is also contained in the class of h-convex functions which were introduced and studied by Varosanec* [24] *by taking suitable choice of the function $h(\cdot)$.*

It is a known fact that the theory of convexity has a deep relationship with the theory of inequalities. Many classical results known in the literature are obtained via convex functions and their various forms. An interesting inequality in this regard is Hermite-Hadamard's inequality which is due to Hermite and Hadamard independently. This result of Hermite and Hadamard provides necessary and sufficient condition for a function to be convex. The classical version of Hermite-Hadamard's inequality given in the following theorem:

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Theorem 1.1. *Let $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int\limits_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}.$$

The left side of Hermite-Hadamard's inequality is estimated by Ostrowski's inequality, which reads as follows:
Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable function on I°, the interior of the interval I, such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'(x)| \leq M$, then, we have

$$\left| f(x) - \frac{1}{b-a}\int\limits_a^b f(u)du \right| \leq \frac{M}{b-a}\left[\frac{(x-a)^2+(b-x)^2}{2}\right].$$

Several authors have shown their special interest in studying this inequality and obtained its several new generalizations, see [2, 3, 8, 14, 15, 23, 24]. Tunc et al. [23] obtained a new version of Hermite-Hadamard's inequality via tgs-convex functions as:

Theorem 1.2. *Let $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a tgs-convex function, then*

$$2f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int\limits_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{6}.$$

For some more details, see [23]. In recent years, several authors have utilized the concepts of quantum calculus for obtaining quantum versions of classical inequalities, see [10, 11, 9, 12, 20, 21]. In this paper, we obtain some quantum versions of certain integral inequalities of Hermite-Hadamard and Ostrowski type via tgs-convex functions. We also discuss some special cases which can be deduced from our main results. Before proceeding further we recall some fundamental concepts of quantum calculus which will be helpful in obtaining our main results.

Let $J = [a,b] \subseteq \mathbb{R}$ be an interval and $0 < q < 1$ be a constant. The q-derivative of a function $f : J \to \mathbb{R}$ at a point $x \in J$ on $[a,b]$ is defined as follows.

Definition 1.2 ([21, 22])**.** *Let $f : J \to \mathbb{R}$ be a continuous function and let $x \in J$. Then q-derivative of f on J at x is defined as*

$$ {}_a\mathcal{D}_q f(x) = \frac{f(x) - f(qx+(1-q)a)}{(1-q)(x-a)}, \quad x \neq a. \tag{2}$$

It is obvious that ${}_a\mathcal{D}_q f(a) = \lim\limits_{x\to a} {}_a\mathcal{D}_q f(x)$.
A function f is q-differentiable on J if ${}_aD_q f(x)$ exists for all $x \in J$. Also if $a = 0$ in (2), then ${}_0\mathcal{D}_q f = {}_a\mathcal{D}_q f$, where ${}_a\mathcal{D}_q$ is the q-derivative of the function f [7] defined as

$$\mathcal{D}_q f(x) = \frac{f(x)-f(qx)}{(1-q)x}$$

.

Definition 1.3 ([21, 22])**.** *Let $f : J \to \mathbb{R}$ be a continuous function. A second-order q-derivative on J, which is denoted as ${}_a\mathcal{D}_q^2 f$, provided ${}_a\mathcal{D}_q f$ is q-differentiable on J is defined as ${}_a\mathcal{D}_q^2 f = {}_a\mathcal{D}_q({}_a\mathcal{D}_q f) : J \to \mathbb{R}$. Similarly higher order q-derivative on J is defined by ${}_a\mathcal{D}_q^n f =: J \to \mathbb{R}$.*

Lemma 1.1 ([21, 22])**.** *Let $\alpha \in \mathbb{R}$, then*

$${}_a\mathcal{D}_q(x-a)^{\alpha} = \Big(\frac{1-q^{\alpha}}{1-q}\Big)(x-a)^{\alpha-1}.$$

Tariboon et al. [21, 22] defined the q-integral as follows:

Definition 1.4 ([21, 22])**.** *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a continuous function. Then q-integral on I is defined as*

$$\int_a^x f(t)\, {}_a\mathrm{d}_q t = (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a), \tag{3}$$

for $x \in I$.

Moreover, if $c \in (a, x)$, then the definite q-integral on J is defined by

$$\begin{aligned}\int_c^x f(t)\, {}_a\mathrm{d}_q t &= \int_a^x f(t)\, {}_a\mathrm{d}_q t - \int_a^c f(t)\, {}_a\mathrm{d}_q t \\ &= (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) \\ &\quad - (1-q)(c-a)\sum_{n=0}^{\infty} q^n f(q^n c + (1-q^n)a).\end{aligned}$$

Theorem 1.3 ([21, 22])**.** *Let $f : I \to \mathbb{R}$ be a continuous function, then*

(1) ${}_a\mathcal{D}_q \int_a^x f(t)\ {}_a\mathrm{d}_q t = f(x)$,

(2) $\int_c^x\ {}_a\mathcal{D}_q f(t)\, {}_a\mathrm{d}_q t = f(x) - f(c)$ *for* $c \in (a, x)$.

Theorem 1.4 ([21, 22])**.** *Let $f, g : I \to \mathbb{R}$ be a continuous functions, $\alpha \in \mathbb{R}$, then $x \in J$*

(1) $\int_a^x [f(t) + g(t)]\ {}_a\mathrm{d}_q t = \int_a^x f(t)\ {}_a\mathrm{d}_q t + \int_a^x g(t)\ {}_a\mathrm{d}_q t$,

(2) $\int_a^x (\alpha f)(t)\ {}_a\mathrm{d}_q t = \alpha \int_a^x f(t)\ {}_a\mathrm{d}_q t$,

(3) $\int_c^x f(t)\ {}_a\mathcal{D}_q g(t)\ {}_a\mathrm{d}_q t = (fg)|_c^x - \int_c^x g(qt + (1-q)a)\, {}_a\mathcal{D}_q f(t)\, {}_a\mathrm{d}_q t$ *for* $c \in (a, x)$.

Lemma 1.2 ([21, 22])**.** *Let $\alpha \in \mathbb{R} \setminus \{-1\}$, then*

$$\int_a^x (t-a)^{\alpha}\ {}_a\mathrm{d}_q t = \Big(\frac{1-q}{1-q^{\alpha+1}}\Big)(x-a)^{\alpha+1}.$$

Proof. Let $f(x) = (x-a)^{\alpha+1}$, $x \in J$ and $\alpha \in \mathbb{R} \setminus \{-1\}$, then by definition, we have

$$\begin{aligned}{}_a\mathcal{D}_q f(x) &= \frac{(x-a)^{\alpha+1} - (qx + (1-q)a - a)^{\alpha+1}}{(1-q)(x-a)} \\ &= \frac{(x-a)^{\alpha+1} - q^{\alpha+1}(x-a)^{\alpha+1}}{(1-q)(x-a)} \\ &= \left(\frac{1-q^{\alpha+1}}{1-q}\right)(x-a)^{\alpha}.\end{aligned} \tag{4}$$

Applying q-integral on J for (4), we obtain the required result. □

We also need following auxiliary results for proving some of our main results.

Lemma 1.3 ([10, 20]). *Let $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on I° (the interior of I) with ${}_a\mathcal{D}_q$ be continuous and integrable on I where $0 < q < 1$, then*

$$\begin{aligned} H_f(a, b; q) &= \frac{1}{b-a} \int_a^b f(x)\, {}_a\mathrm{d}_q x - \frac{qf(a) + f(b)}{1+q} \\ &= \frac{q(b-a)}{1+q} \int_0^1 (1 - (1+q)t)\, {}_a\mathcal{D}_q f((1-t)a + tb)\, {}_0\mathrm{d}_q t. \end{aligned}$$

Lemma 1.4 ([12]). *Let $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on I° (the interior of I) with ${}_a\mathcal{D}_q$ be continuous and integrable on I where $0 < q < 1$, then*

$$\begin{aligned} K_f(a, b; q) &= f(x) - \frac{1}{b-a} \int_a^b f(u)\, {}_a\mathrm{d}_q u \\ &= \frac{q(x-a)^2}{b-a} \int_0^1 t\, {}_a\mathcal{D}_q f(tx + (1-t)a)\, {}_0\mathrm{d}_q t + \frac{q(b-x)^2}{b-a} \int_0^1 t\, {}_a\mathcal{D}_q f(tx + (1-t)b)\, {}_0\mathrm{d}_q t. \end{aligned}$$

2. Main results

In this section, we discuss our main results.

Theorem 2.1. *Let $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$ be tgs-convex function, then*

$$2f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x)\, {}_a\mathrm{d}_q x \le \psi_1(q)[f(a) + f(b)],$$

where

$$\psi_1(q) = \frac{1}{1+q} - \frac{1}{1+q+q^2}. \tag{5}$$

Proof. Since it is given that f is a tgs-convex function, then

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{ta + (1-t)b + (1-t)a + tb}{2}\right) \\ &\le \frac{1}{4}[f(ta + (1-t)b) + f((1-t)a + tb)]. \end{aligned}$$

q-integrating above inequality with respect to t on $[0, 1]$, we have

$$2f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x)\, {}_a\mathrm{d}_q x. \tag{6}$$

Also

$$f(ta + (1-t)b) \le t(1-t)[f(a) + f(b)].$$

q-integrating above inequality with respect to t on $[0, 1]$, we have

$$\frac{1}{b-a} \int_a^b f(x)\, {}_a\mathrm{d}_q x \le \left(\frac{1}{1+q} - \frac{1}{1+q+q^2}\right)[f(a) + f(b)]. \tag{7}$$

On summation of inequalities (6) and (7), we get the required result. □

Remark 2.1. *If $q \to 1$ in Theorem 2.1, we get Theorem 2.1* [23].

Our next result is q-Hermite-Hadamard's inequality via product of two tgs-convex functions.

Theorem 2.2. *Let $f, g : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be two tgs-convex functions, then*

I. $8f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) - \psi_2(q)[M(a,b) + N(a,b)] \le \frac{1}{b-a} \int_a^b f(x)g(x)\, {}_a\mathrm{d}_q x,$

II. $\frac{1}{b-a} \int_a^b f(x)g(x)\, {}_a\mathrm{d}_q x \le \psi_2(q)[M(a,b) + N(a,b)],$

where

$$M(a,b) = f(a)g(a) + f(b)g(b); \tag{8}$$

$$N(a,b) = f(a)g(b) + f(b)g(a); \tag{9}$$

and

$$\psi_2(q) = \frac{1}{1+q+q^2} + \frac{1}{1+q+q^2+q^3+q^4} - \frac{2}{1+q+q^2+q^3}. \tag{10}$$

Proof. **I.** Since it is given that f and g are tgs-convex functions, then

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\
& \le \frac{1}{16}[f(ta+(1-t)b) + f((1-t)a+tb)][g(ta+(1-t)b) + g((1-t)a+tb)] \\
& = \frac{1}{16}\left\{f(ta+(1-t)b)g(ta+(1-t)b) + f((1-t)a+tb)g((1-t)a+tb)\right. \\
& \quad \left. + f((1-t)a+tb)g(ta+(1-t)b) + f(ta+(1-t)b)g((1-t)a+tb)\right\}.
\end{aligned}$$

q-integrating both sides of above inequality with respect to t on $[0,1]$, we have

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\
& \le \frac{1}{16}\left\{\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)\, {}_0\mathrm{d}_q t + \int_0^1 f((1-t)a+tb)g((1-t)a+tb)\, {}_0\mathrm{d}_q t\right. \\
& \quad \left. + \int_0^1 f((1-t)a+tb)g(ta+(1-t)b)\, {}_0\mathrm{d}_q t + \int_0^1 f(ta+(1-t)b)g((1-t)a+tb)\, {}_0\mathrm{d}_q t\right\} \\
& = \frac{1}{16}\left\{\frac{2}{b-a}\int_a^b f(x)g(x)\, {}_a\mathrm{d}_q x + 2\int_0^1 t^2(1-t)^2[f(a)+f(b)][g(a)+g(b)]\, {}_0\mathrm{d}_q t\right\} \\
& = \frac{1}{8}\left\{\frac{1}{b-a}\int_a^b f(x)g(x)\, {}_a\mathrm{d}_q x + \left(\frac{1}{1+q+q^2} + \frac{1}{1+q+q^2+q^3+q^4} - \frac{2}{1+q+q^2+q^3}\right)\right. \\
& \qquad \left. \times [f(a)+f(b)][g(a)+g(b)]\right\} \\
& = \frac{1}{8}\left\{\frac{1}{b-a}\int_a^b f(x)g(x)\, {}_a\mathrm{d}_q x + \left(\frac{1}{1+q+q^2} + \frac{1}{1+q+q^2+q^3+q^4} - \frac{2}{1+q+q^2+q^3}\right)\right. \\
& \qquad \left. \times [M(a,b)+N(a,b)]\right\}.
\end{aligned}$$

This completes the proof of first part.

II. Now we prove second part of the theorem. Using the hypothesis of the theorem that f and g are tgs-convex functions, we have

$$f(ta+(1-t)b)g(ta+(1-t)b) \leq t^2(1-t)^2[f(a)+f(b)][g(a)+g(b)].$$

q-integrating both sides of above inequality with respect to t on $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)\,{}_0\mathrm{d}_q t \\
&\leq [f(a)+f(b)][g(a)+g(b)]\int_0^1 t^2(1-t)^2\,{}_0\mathrm{d}_q t \\
&= [M(a,b)+N(a,b)]\left(\frac{1}{1+q+q^2}+\frac{1}{1+q+q^2+q^3+q^4}-\frac{2}{1+q+q^2+q^3}\right).
\end{aligned}$$

This completes the proof of second part. $\square$

Remark 2.2. *Note that when $q \to 1$ Theorem 2.2 reduces to previously known results, see* [23].

Theorem 2.3. *Let $f, g : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be two tgs-convex functions, then*

$$\begin{aligned}
&\frac{\psi_2^{-1}(q)}{2(b-a)}\int_a^b\int_a^b\int_0^1 f(tx+(1-t)y)g(tx+(1-t)y)\,{}_0\mathrm{d}_q t\,{}_a\mathrm{d}_q y\,{}_a\mathrm{d}_q x \\
&\leq \frac{1}{b-a}\int_a^b f(x)g(x)\,{}_a\mathrm{d}_q x + \psi_1^2[M(a,b)+N(a,b)],
\end{aligned}$$

where $M(a,b)$, $N(a,b)$, $\psi_1(q)$ and $\psi_2(q)$ are given by (8), (9), (5) and (10) respectively.

Proof. Since it is given that f and g are tgs-convex functions, then

$$f(tx+(1-t)y)g(tx+(1-t)y) \leq t^2(1-t)^2[f(x)+f(y)][g(x)+g(y)].$$

q-integrating both sides of above inequality with respect to t on the interval $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 f(tx+(1-t)y)g(tx+(1-t)y)\,{}_0\mathrm{d}_q t \\
&\leq \int_0^1 t^2(1-t)^2[f(x)+f(y)][g(x)+g(y)]\,{}_0\mathrm{d}_q t \\
&= \left[\frac{1}{1+q+q^2}+\frac{1}{1+q+q^2+q^3+q^4}-\frac{2}{1+q+q^2+q^3}\right][f(x)+f(y)][g(x)+g(y)].
\end{aligned}$$

Now again q-integrating both sides of above inequality on $[a,b]\times[a,b]$, we have

$$\int_a^b\int_a^b\int_0^1 f(tx+(1-t)y)g(tx+(1-t)y)\,{}_0\mathrm{d}_qt\,{}_a\mathrm{d}_qy\,{}_a\mathrm{d}_qx$$
$$\leq\left[\frac{1}{1+q+q^2}+\frac{1}{1+q+q^2+q^3+q^4}-\frac{2}{1+q+q^2+q^3}\right]$$
$$\times\int_a^b\int_a^b[f(x)+f(y)][g(x)+g(y)]\,{}_a\mathrm{d}_qy\,{}_a\mathrm{d}_qx$$
$$=\left[\frac{1}{1+q+q^2}+\frac{1}{1+q+q^2+q^3+q^4}-\frac{2}{1+q+q^2+q^3}\right]$$
$$\times\int_a^b\int_a^b[f(x)g(x)+f(y)g(y)+f(x)g(y)+f(y)g(x)]\,{}_a\mathrm{d}_qy\,{}_a\mathrm{d}_qx$$
$$=\left[\frac{1}{1+q+q^2}+\frac{1}{1+q+q^2+q^3+q^4}-\frac{2}{1+q+q^2+q^3}\right]$$
$$\times\left[\int_a^b\int_a^b[f(x)g(x)+f(y)g(y)]\,{}_a\mathrm{d}_qy\,{}_a\mathrm{d}_qx\right.$$
$$\left.+\int_a^b f(x)\,{}_a\mathrm{d}_qx\int_a^b g(y)\,{}_a\mathrm{d}_qy+\int_a^b f(y)\,{}_a\mathrm{d}_qy\int_a^b g(x)\,{}_a\mathrm{d}_qx\right].$$

Using Theorem 2.1, we have

$$\int_a^b\int_a^b\int_0^1 f(tx+(1-t)y)g(tx+(1-t)y)\,{}_0\mathrm{d}_qt\,{}_a\mathrm{d}_qy\,{}_a\mathrm{d}_qx$$
$$\leq\left[\frac{1}{1+q+q^2}+\frac{1}{1+q+q^2+q^3+q^4}-\frac{2}{1+q+q^2+q^3}\right]$$
$$\times\left[2(b-a)\int_a^b f(x)g(x)\,{}_a\mathrm{d}_qx+2(b-a)^2\left[\frac{1}{1+q}-\frac{1}{1+q+q^2}\right]^2[M(a,b)+N(a,b)]\right]$$
$$=2\psi_2(q)\left[(b-a)\int_a^b f(x)g(x)\,{}_a\mathrm{d}_qx+(b-a)^2\psi_1^2(q)[M(a,b)+N(a,b)]\right].$$

Multiplying both sides of of above inequality by $\frac{1}{(b-a)^2}$ completes the proof. □

Note that when $t\to1$ in Theorem 2.3, we get Theorem 2.4 [23].
Now we prove some q-Hermite-Hadamard type inequalities via q-differentiable tgs-convex functions.

Theorem 2.4. *Let $f:I=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a q-differentiable function on I° (the interior of I) with ${}_a\mathcal{D}_q$ be continuous and integrable on I where $0<q<1$. If $|\,{}_a\mathcal{D}_q|$ is tgs-convex function, then*

$$|H_f(a,b;q)|\leq\psi_3(q)(b-a)\{|\,{}_a\mathcal{D}_qf(a)|+|\,{}_a\mathcal{D}_qf(b)|\},$$

where

$$\psi_3(q) = \frac{q^4(1-q)}{(1+q)^4(1+q+q^2)(1+q+q^2+q^3)}. \tag{11}$$

Proof. Utilizing Lemma 1.3, property of modulus and the hypothesis of the theorem, we have

$$\begin{aligned}
&|H_f(a,b;q)| \\
&= \left| \frac{q(b-a)}{1+q} \int_0^1 (1-(1+q)t)\, {}_a\mathcal{D}_q f((1-t)a+tb)\ {}_0\mathrm{d}_q t \right| \\
&\le \frac{q(b-a)}{1+q} \int_0^1 |1-(1+q)t| |\, {}_a\mathcal{D}_q f((1-t)a+tb)|\, {}_0\mathrm{d}_q t \\
&\le \frac{q(b-a)}{1+q} \int_0^1 |1-(1+q)t| [t(1-t)\{|\, {}_a\mathcal{D}_q f(a)| + |\, {}_a\mathcal{D}_q f(b)|\}]\, {}_0\mathrm{d}_q t \\
&= \frac{q^4(1-q)(b-a)}{(1+q)^4(1+q+q^2)(1+q+q^2+q^3)} \{|\, {}_a\mathcal{D}_q f(a)| + |\, {}_a\mathcal{D}_q f(b)|\}.
\end{aligned}$$

This completes the proof. □

Theorem 2.5. *Let $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on I° (the interior of I) with ${}_a\mathcal{D}_q$ be continuous and integrable on I where $0 < q < 1$. If $|\, {}_a\mathcal{D}_q|^r$, $r \ge 1$ is tgs-convex function, then*

$$|H_f(a,b;q)| \le \frac{q(b-a)}{1+q} \left(\frac{2q}{(1+q)^2} \right)^{1-\frac{1}{r}} \left(\psi_4(q)\{|\, {}_a\mathcal{D}_q f(a)|^r + |\, {}_a\mathcal{D}_q f(b)|^r\} \right)^{\frac{1}{r}},$$

where

$$\psi_4(q) = \frac{q^3(1-q)}{(1+q)^3(1+q+q^2)(1+q+q^2+q^3)}. \tag{12}$$

Proof. Utilizing Lemma 1.3, property of modulus, power mean inequality and the hypothesis of the theorem, we have

$$\begin{aligned}
&|H_f(a,b;q)| \\
&= \left| \frac{q(b-a)}{1+q} \int_0^1 (1-(1+q)t)\, {}_a\mathcal{D}_q f((1-t)a+tb)\ {}_0\mathrm{d}_q t \right| \\
&\le \frac{q(b-a)}{1+q} \left(\int_0^1 |1-(1+q)t|\, {}_0\mathrm{d}_q t \right)^{1-\frac{1}{r}} \left(\int_0^1 (1-(1+q)t) |\, {}_a\mathcal{D}_q f((1-t)a+tb)|^r\, {}_0\mathrm{d}_q t \right)^{\frac{1}{r}} \\
&\le \frac{q(b-a)}{1+q} \left(\frac{2q}{(1+q)^2} \right)^{1-\frac{1}{r}} \left(\int_0^1 |1-(1+q)t| [t(1-t)\{|\, {}_a\mathcal{D}_q f(a)|^r + |\, {}_a\mathcal{D}_q f(b)|^r\}]\, {}_0\mathrm{d}_q t \right)^{\frac{1}{r}} \\
&= \frac{q(b-a)}{1+q} \left(\frac{2q}{(1+q)^2} \right)^{1-\frac{1}{r}} \left(\frac{q^3(1-q)}{(1+q)^3(1+q+q^2)(1+q+q^2+q^3)} \{|\, {}_a\mathcal{D}_q f(a)|^r + |\, {}_a\mathcal{D}_q f(b)|^r\} \right)^{\frac{1}{r}} \\
&= \frac{q(b-a)}{1+q} \left(\frac{2q}{(1+q)^2} \right)^{1-\frac{1}{r}} \left(\psi_4(q)\{|\, {}_a\mathcal{D}_q f(a)|^r + |\, {}_a\mathcal{D}_q f(b)|^r\} \right)^{\frac{1}{r}}.
\end{aligned}$$

This completes the proof. □

Next we prove some q-Ostrowski type inequalities via q-differentiable tgs-convex functions.

Theorem 2.6. *Let $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on I° (the interior of I) with ${}_a\mathcal{D}_q$ be continuous and integrable on I where $0 < q < 1$. If $|{}_a\mathcal{D}_q f|$ is tgs-convex function and $|{}_a\mathcal{D}_q f(x)| \leq M$, then, we have*

$$|K_f(a, b; q)| \leq \frac{2\psi_5(q)Mq[(x-a)^2 + (b-x)^2]}{b-a},$$

where

$$\psi_5(q) = \frac{q^3}{(1+q+q^2)(1+q+q^2+q^3)}. \tag{13}$$

Proof. Using Lemma 1.4, property of the modulus and hypothesis of the theorem, we have

$$\begin{aligned}
&|K_f(a, b; q)| \\
&= \left| \frac{q(x-a)^2}{b-a} \int_0^1 t\,{}_a\mathcal{D}_q f(tx + (1-t)a)\,{}_0\mathrm{d}_q t + \frac{q(b-x)^2}{b-a} \int_0^1 t\,{}_a\mathcal{D}_q f(tx + (1-t)b)\,{}_0\mathrm{d}_q t \right| \\
&\leq \frac{q(x-a)^2}{b-a} \int_0^1 t|{}_a\mathcal{D}_q f(tx + (1-t)a)|\,{}_0\mathrm{d}_q t + \frac{q(b-x)^2}{b-a} \int_0^1 t|{}_a\mathcal{D}_q f(tx + (1-t)b)|\,{}_0\mathrm{d}_q t \\
&\leq \frac{q(x-a)^2}{b-a} \int_0^1 t[t(1-t)\{|{}_a\mathcal{D}_q(x)| + |{}_a\mathcal{D}_q(a)|\}]\,{}_0\mathrm{d}_q t \\
&\qquad + \frac{q(b-x)^2}{b-a} \int_0^1 t[t(1-t)\{|{}_a\mathcal{D}_q(x)| + |{}_a\mathcal{D}_q(b)|\}]\,{}_0\mathrm{d}_q t \\
&\leq \frac{2Mq[(x-a)^2 + (b-x)^2]}{b-a} \left\{ \int_0^1 t^2(1-t)\,{}_0\mathrm{d}_q t \right\} \\
&= \frac{2Mq[(x-a)^2 + (b-x)^2]}{b-a} \left\{ \frac{q^3}{(1+q+q^2)(1+q+q^2+q^3)} \right\}.
\end{aligned}$$

This completes the proof. □

Theorem 2.7. *Let $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on I° (the interior of I) with ${}_a\mathcal{D}_q$ be continuous and integrable on I where $0 < q < 1$. If $|{}_a\mathcal{D}_q f|^r$ is tgs-convex function and $|{}_a\mathcal{D}_q f(x)| \leq M$, then for $p, r > 1$, $\frac{1}{p} + \frac{1}{r} = 1$, we have*

$$|K_f(a, b; q)| \leq \frac{2\psi_1(q)Mq[(x-a)^2 + (b-x)^2]}{(b-a)} \Big(\frac{1-q}{1-q^{p+1}} \Big)^{\frac{1}{p}},$$

where $\psi_1(q)$ is given by (5).

Proof. Using Lemma 1.4, Holder's inequality and the hypothesis of the theorem, we have

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,{}_a\mathrm{d}_q u\right| \\
&=\left|\frac{q(x-a)^2}{b-a}\int_0^1 t\,{}_a\mathcal{D}_q f(tx+(1-t)a)\,{}_0\mathrm{d}_q t+\frac{q(b-x)^2}{b-a}\int_0^1 t\,{}_a\mathcal{D}_q f(tx+(1-t)b)\,{}_0\mathrm{d}_q t\right| \\
&\le \frac{q(x-a)^2}{b-a}\Big(\int_0^1 t^p\,{}_0\mathrm{d}_q t\Big)^{\frac{1}{p}}\Big(\int_0^1 |\,{}_a\mathcal{D}_q f(tx+(1-t)a)|^r\,{}_0\mathrm{d}_q t\Big)^{\frac{1}{r}} \\
&\qquad+\frac{q(b-x)^2}{b-a}\Big(\int_0^1 t^p\,{}_0\mathrm{d}_q t\Big)^{\frac{1}{p}}\Big(\int_0^1 |\,{}_a\mathcal{D}_q f(tx+(1-t)b)|^r\,{}_0\mathrm{d}_q t\Big)^{\frac{1}{r}} \\
&\le \frac{q(x-a)^2}{b-a}\Big(\frac{1-q}{1-q^{p+1}}\Big)^{\frac{1}{p}}\Big(\int_0^1 [t(1-t)\{|\,{}_a\mathcal{D}_q f(x)|^r+|\,{}_a\mathcal{D}_q f(a)|^r\}\,{}_0\mathrm{d}_q t\Big)^{\frac{1}{r}} \\
&\qquad+\frac{q(b-x)^2}{b-a}\Big(\frac{1-q}{1-q^{p+1}}\Big)^{\frac{1}{p}}\Big(\int_0^1 [t(1-t)\{|\,{}_a\mathcal{D}_q f(x)|^r+|\,{}_a\mathcal{D}_q f(b)|^r\}\,{}_0\mathrm{d}_q t\Big)^{\frac{1}{r}} \\
&\le \frac{2qM[(x-a)^2+(b-x)^2]}{(b-a)}\Big(\frac{1-q}{1-q^{p+1}}\Big)^{\frac{1}{p}}\left\{\frac{q^2}{(1+q)(1+q+q^2)}\right\}^{\frac{1}{r}}.
\end{aligned}$$

This completes the proof. □

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